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# CALCULATION OF A LAMINAR BOUNDARY LAYER ON A ROTATING POROUS DISK


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UDC 532.526.75


Averaging of noninertial terms over the boundary-layer section in the equations of motion is used to study the effect of suction and injection on the hydrodynamic flow near a rotating disk.


Control of a boundary layer by suction or injection of one or the other liquid through a porous disk is a technique which is widely used in technology at present [1, 2].

Solution of the laminar boundary-layer equations with consideration of the effect of flow through the porous surface of the body over which the flow takes place is a complex problem which in most cases is solved numerically [3]. However, in a number of technical applications there is a need for analytical expressions for the hydrodynamic flow profiles and boundary-layer thicknesses [4, 5]. In a number of problems the Slezkin–Targa method has been used for this purpose. This method consists of averaging the nonlinear terms in the equations of motion over the boundary-layer thickness [6, 7].

In the present study a modification of this method will be used to calculate the laminar boundary layer in a viscous incompressible liquid on a rotating porous disk of infinite radius in the presence of uniform suction or injection of a liquid with the same physical properties as the main liquid.

In the notation generally used the equations of the spatial boundary layer on a rotating disk have the form [6]:

$$U\frac{\partial U}{\partial r}+W\frac{\partial U}{\partial z}-\frac{V^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\nu\frac{\partial^2 U}{\partial z^2}, \tag{1}$$


Moscow Engineering Physics Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 49, No. 6, pp. 1022-1026, December, 1985. Original article submitted October 3, 1984.

$$U\frac{\partial V}{\partial r}+W\frac{\partial V}{\partial z}+\frac{UV}{r}=\nu\frac{\partial^2 V}{\partial z^2}, \qquad (2)$$

$$\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{\partial W}{\partial z}=0. \qquad (3)$$

System (1)-(3) must be integrated with boundary conditions

$$\begin{gathered} z=0 \quad U=0, \quad V=\omega r, \quad W=-k; \\ z\to\infty \quad U\to 0, \quad V\to 0. \end{gathered} \qquad (4)$$

A positive k value corresponds to suction in the boundary layer, and a negative to injection.

Assuming in Eqs. (1) and (2) that

$$W(r,\ z)=-k+W_0(r,\ z) \qquad (5)$$

and separating the inertial terms into linear and nonlinear components [8], we obtain:

$$U\frac{\partial U}{\partial r}+W_0\frac{\partial U}{\partial z}-\frac{V^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\nu\frac{\partial^2 U}{\partial z^2}+k\frac{\partial U}{\partial z}, \qquad (6)$$

$$U\frac{\partial V}{\partial r}+W_0\frac{\partial V}{\partial z}+\frac{UV}{r}=\nu\frac{\partial^2 V}{\partial z^2}+k\frac{\partial V}{\partial z}, \qquad (7)$$

$$\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{\partial W_0}{\partial z}=0. \qquad (8)$$

Taking U = rF(z), V = rG(z), considering the condition $\frac{1}{\rho}\frac{\partial P}{\partial r}=0$, and replacing the inertial terms by their mean value over the boundary-layer section, we have:

$$\frac{1}{\nu\delta}\int_0^\delta 3F^2dz-\frac{1}{\nu\delta}\int_0^\delta G^2dz=\frac{d^2F}{dz^2}+\frac{1}{h}\frac{dF}{dz}, \qquad (9)$$

$$\frac{1}{\nu\delta}\int_0^\delta 4FGdz=\frac{d^2G}{dz^2}+\frac{1}{h}\frac{dG}{dz}, \qquad (10)$$

where $h = \nu/k$.

Integrating Eqs. (9), (10) and using the approximate boundary conditions:

$$\begin{gathered} z=0 \quad F=0, \quad G=\omega; \qquad (11)\\ z=\delta \quad F=0, \quad G=0, \quad \frac{dG}{dz}=0, \end{gathered}$$

we find

$$F=-\frac{A\delta h\left[\exp\left(-\frac{z}{h}\right)-1\right]}{\exp\left(\frac{\delta}{h}\right)-1}+Azh, \qquad (12)$$

$$G=\frac{\omega}{\exp\left(\frac{\delta}{h}\right)-\frac{\delta}{h}-1}\left[\exp\left(-\frac{z}{h}+\frac{\delta}{h}\right)-\left(\frac{\delta}{h}-\frac{z}{h}\right)-1\right]. \qquad (13)$$

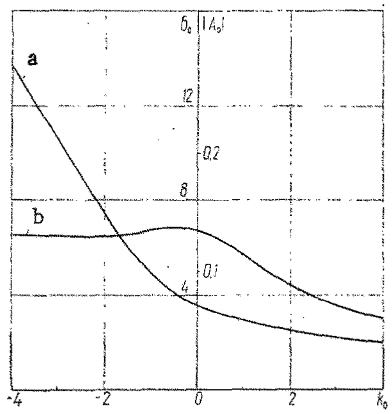


Fig. 1

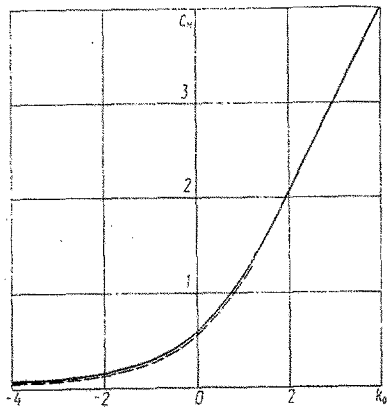


Fig. 2

Fig. 1. Boundary-layer thickness $\delta_0$ (a) and parameter $A_0$ (b) vs suction (injection) intensity $k_0$. All quantities dimensionless.

Fig. 2. Coefficient of friction momentum $c_M$ (dimensionless) vs suction (injection) intensity $k_0$. Dashes, data of [9].

To determine the unknowns A and $\delta$ the following system of algebraic equations must be solved:

$$A_0 = A_0^2\delta_0^4\left(\frac{3}{x^2}Q + \frac{6}{x^4}R + \frac{1}{x^2}\right) - \frac{\exp(-2x)}{[1-\exp(-x) - x\exp(-x)]^2} \times \tag{14}$$

$$\times\left\{\frac{1}{2x}[\exp(2x) - 1] + 1 + x + \frac{x^2}{3} - 2\exp(x)\right\},$$

$$A_0 = \frac{1}{4\delta_0^4 D}, \tag{15}$$

where

$$Q = \frac{1 + \frac{1}{2x}[-3 - \exp(-2x) + 4\exp(-x)]}{1 - 2\exp(-x) + \exp(-2x)},$$

$$R = \frac{1 - (x+1)\exp(-x) - \frac{x^2}{2}}{1 - \exp(-x)},$$

$$D = \frac{\frac{1}{x} + \frac{x}{2} - \frac{1}{2x}[\exp(x) + \exp(-x)]}{x^3[1 - \exp(-x)]} - \frac{1}{x^5}[1 - \exp(x)] - \frac{1}{x^4} - \frac{2}{x^3} - \frac{1}{6x^2},$$

$$x = k_0\delta_0, \quad \delta_0 = \delta\left(\frac{\omega}{\nu}\right)^{1/2}, \quad k_0 = k(\nu\omega)^{-1/2}, \quad A_0 = A\frac{\nu}{\omega^2}.$$

In the limiting case of an impermeable disk ($k_0 = 0$) the solution coincides with the data of [6]. The dependences of the dimensionless boundary-layer thickness $\delta_0$ and the quantity $A_0$, which characterizes the velocity of radial motion of the medium near the disk surface, on the parameter $k_0$ are shown in Fig. 1. It is evident that suction and especially injection have a significant effect on boundary-layer thickness. The dependence of $A_0$ on $k_0$ is caused by the characteristic change in the values of the corresponding terms in Eqs. (1), (2), describing the balance of viscous and "inertial" forces. It is obvious that centrifugal forces tend

TABLE 1. Comparison of Friction Momentum Coefficients for Various Suction Intensities

| Friction momentum coefficients | $k_0$ | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 0,5 | 1,0 | 2,0 | 3,0 | 4,0 |
| $c_M$ | 0,572 | 0,845 | 1,197 | 2,064 | 3,024 | 4,011 |
| $c_M^*$ | 0,616 | — | 1,175 | 2,041 | 3,012 | 4,005 |

to remove liquid in the boundary layer to the periphery, while viscous forces inhibit this process. In the case of suction the main cause for the decrease in $A_0$ with increase in $k_0$ is the increase in radial viscous friction forces near the disk surface. Injection leads to thickening of the boundary layer, which creates a tendency toward increase in the intensity of the radial flow. However, there is then an increase in the "inertial" force caused by change in radial velocity with radial coordinate, which is directed opposite to the centrifugal force. The latter is the cause of the characteristic behavior of $A_0$.

Using Eq. (13) for the azimuthal velocity component, we calculate the moments of the friction forces $M_0$ acting on one side of the surface of a disk of radius $R_0$:

$$M_0 = 2\pi\rho\nu \int\limits_0^{R_0} r^2 \left| \frac{\partial V}{\partial z} \right|_{z=0} dr = \frac{\pi\rho R_0^4 (\nu\omega^3)^{1/2} (x[\exp(-x) - 1]}{2\delta_0 [1 - x + 1) \exp(-x)]}. \tag{16}$$

The dependence of the coefficient of friction momentum

$$c_M = \frac{2M_0}{\pi\rho R_0^4 (\nu\omega^3)^{1/2}} \tag{17}$$

on the parameter $k_0$ is shown in Fig. 2. The dashed curve is the result of a calculation performed in [9]. Table 1 presents a comparison of $c_M$ values found by different methods. Therein $c_M^*$ is the coefficient determined in [6] by merging of an approximate solution for low suctions with an exact solution for high suction.

We note the satisfactory agreement of the values obtained by the different methods, and stress the simplicity and convenience of the approach used in the present study for analysis of the boundary-layer equations. The analytical expressions found for the hydrodynamic flow characteristics are valid for values of the parameter $k_0$ at which the boundary layer is stable and effects involving its retraction from the disk surface do not develop.

## NOTATION

U, V, W, radial, circular, and axial velocities; r, z, radial and axial coordinates; P, pressure; $\rho$, density; $\nu$, kinematic viscosity of medium; k, suction (injection) parameter; $\delta$, boundary-layer thickness; A, parameter characterizing radial velocity of the medium motion; $M_0$, moment of friction forces; $c_M$, friction momentum coefficient; $R_0$, disk radius; $\omega$, angular velocity of disk rotation.

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# STRUCTURE OF THE SINGULAR TERMS IN THE FREE ENERGY CORRECTLY REPRODUCING THE NONASYMPTOTIC CORRECTIONS TO THE THERMODYNAMIC FUNCTIONS

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UDC 536.71

The method for selecting the structure of the singular terms in the expression for the free energy correctly reproducing the nonasymptotic components of the thermodynamic functions is examined.

The problem of describing a wide neighborhood of the critical point with the help of unified nonanalytical equations of state is solved in [1-13]. In these works, the starting thermodynamic functions — the internal energy u(ρ, T) [1-6], the enthalpy i(p, T) [7, 8], the chemical potential μ(p, T) [9], the isochoric heat capacity $C_V$(ρ, T) [10], and the Helmholtz free energy F(ρ, T) [11-13] — are represented in the form of two terms: an irregular term satisfying a power law of the scaling theory (ST) and a regular function describing the characteristic features of the thermodynamic surface in the region of low densities and pressures.

The singular components of the equations of state [1-13] enable describing qualitatively correctly, i.e., in accordance with the requirements of the ST, the behavior of the thermodynamic surface only in an asymptotic neighborhood of the critical point: $|\Delta\rho| \leqslant 0{,}06,\ \tau \leqslant 0{,}01$. At the same time, according to [14, 15], in describing the properties of pure substances (in our case liquid—vapor systems) auxiliary nonanalytical terms, taking into account the next approximations of ST, must be included in the structure of the equations of state. These correction terms are calculated in [16] by the ε expansion method up to terms of order $\varepsilon^2$. In accordance with the results of [16, 17], the behavior of a number of thermodynamic functions on characteristic lines of the thermodynamic surface is described in the critical region by the following power laws:

$$K_T(\rho_c\ T) = \Phi_0\tau^{-\gamma} + \Phi_1\tau^{-\gamma+\Delta}, \tag{1}$$

$$C_v(\rho_c,\ T) = \Gamma_0\tau^{-\alpha} + \Gamma_1\tau^{-\alpha+\Delta} + \Gamma_2, \tag{2}$$

$$p(\rho,\ T_c) - p(\rho_c,\ T_c) = P_0\Delta\rho\,|\Delta\rho|^{\delta-1} + P_1\Delta\rho\,|\Delta\rho|^{\delta-1+\frac{\Delta}{\beta}}, \tag{3}$$

$$\mu(\rho,\ T_c) - \mu(\rho_c,\ T_c) = R_0\Delta\rho\,|\Delta\rho|^{\delta-1} + R_1\Delta\rho\,|\Delta\rho|^{\delta-1+\frac{\Delta}{\beta}}. \tag{4}$$

Here $\Phi_0$, $\Gamma_0$, $P_0$, $R_0$ are the constant coefficients in front of the asymptotic terms of the expressions (1)-(4), $\Phi_1$, $\Gamma_1$, $P_1$, $R_1$ are the constant coefficients in front of the nonasymptotic correction terms in the expressions (1)-(4).

The structure of the nonasymptotic terms has now been established only for the scale equations of state in parametric form:

Leningrad Technological Institute of the Refrigeration Industry. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 49, No. 6, pp. 1027-1033, December, 1985. Original article submitted September 29, 1984.

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